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Remark on hyperstability of the general linear equation

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Abstract. We present a result concerning the hyperstability of the general linear equation. Namely, we show that a function satisfying the equation approximately must be actually a solution to it.

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Throughout the paper $\mathbb{F}$, $\mathbb{K}$ denote the fields of real or complex numbers. Let X, Y be normed spaces over $\mathbb{F}$, $\mathbb{K}$, respectively. The aim of the paper is to give a hyperstability result for the general linear equation

$$g(ax + by) = Ag(x) + Bg(y), \tag{1}$$

where $g\colon X \to Y$ and $a, b \in \mathbb{F}\backslash\{0\}$, $A, B \in \mathbb{K}$. We see that for $a = b = A = B = 1$ in (1) we get the Cauchy equation while the Jensen equation corresponds to $a = b = A = B = \frac{1}{2}$. The general linear equation has been studied by many authors, see for example [6,8,10–12,14].

Probably the first hyperstability result was published in [3] and concerned ring homomorphisms. However the term hyperstability was used for the first time in [13].

We recall the most classical results concerning the Hyers-Ulam stability of the Cauchy equation

$$f(x + y) = f(x) + f(y), \qquad x, y \in X, \tag{2}$$

which were the motivation to write the paper.

If X is a normed space, Y is a Banach space, $c \geq 0$, p is a real number such that $0 < p < 1$ and $f\colon X \to Y$ satisfies the inequality

$$\|f(x + y) - f(x) - f(y)\| \leq c(\|x\|^p + \|y\|^p), \qquad x, y \in X, \tag{3}$$

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then there exists a unique solution $T\colon X \to Y$ *of* (2) *with*

$$\|f(x) - T(x)\| \le \frac{c\|x\|^p}{|1 - 2^{p-1}|}, \qquad x \in X.$$

It is due to Aoki [1] (see also [15]). Later Gajda [7] obtained this result for $p > 1$ and Rassias [16] for $p < 0$. Moreover, in [7] we can find an example showing that an analogous result for $p = 1$ is not true. For $p = 0$ it is the first result of stability proved by Hyers [9]. Now, it is known that for $p < 0$ we have the hyperstability result, that is f satisfying (3) must be additive for $x, y \in X\backslash\{0\}$ (see [4]).

The method of the proof used in the main theorem is based on a fixed point result that can be derived from [5] (Theorem 1). To present it we need the following three hypothesis:

(H1) U is a nonempty set, Y is a Banach space, $f_1, \ldots, f_k\colon U \to U$ and $L_1, \ldots, L_k\colon U \to \mathbb{R}_+$ are given.

(H2) $\mathcal{T}\colon Y^U \to Y^U$ is an operator satisfying the inequality

$$\big\|\mathcal{T}\xi(x) - \mathcal{T}\mu(x)\big\| \le \sum_{i=1}^{k} L_i(x)\big\|\xi(f_i(x)) - \mu(f_i(x))\big\|,$$
$$\xi, \mu \in Y^U, x \in U.$$

(H3) $\Lambda\colon {\mathbb{R}_+}^U \to {\mathbb{R}_+}^U$ is defined by

$$\Lambda\delta(x) := \sum_{i=1}^{k} L_i(x)\delta(f_i(x)), \qquad \delta \in {\mathbb{R}_+}^U, x \in U.$$

Now we are in a position to present the above mentioned fixed point theorem.

Theorem 1. *Let hypotheses* (H1)–(H3) *be valid and functions* $\varepsilon\colon U \to \mathbb{R}_+$ *and* $\varphi\colon U \to Y$ *fulfil the following two conditions*

$$\big\|\mathcal{T}\varphi(x) - \varphi(x)\big\| \le \varepsilon(x), \qquad x \in U,$$
$$\varepsilon^*(x) := \sum_{n=0}^{\infty} \Lambda^n\varepsilon(x) < \infty, \qquad x \in U.$$

Then there exists a unique fixed point ψ *of* $\mathcal{T}$ *with*

$$\|\varphi(x) - \psi(x)\| \le \varepsilon^*(x), \qquad x \in U.$$

Moreover

$$\psi(x) := \lim_{n\to\infty} \mathcal{T}^n\varphi(x), \qquad x \in U.$$

Theorem 2. *Let* X *be a normed space over a field* $\mathbb{F}$, Y *be a Banach space over* $\mathbb{K}$, $a, b \in \mathbb{F}\backslash\{0\}$, $A, B \in \mathbb{K}$, $c \ge 0$, $p < 0$ *and* $g\colon X \to Y$ *satisfy*

$$\|g(ax + by) - Ag(x) - Bg(y)\| \le c(\|x\|^p + \|y\|^p), \qquad x, y \in X\backslash\{0\}. \tag{4}$$

Then g satisfies the equation

$$g(ax+by) = Ag(x) + Bg(y), \qquad x, y \in X\backslash\{0\}. \tag{5}$$

Proof. Replacing x by $\frac{1}{a}(m+1)x$ and y by $-\frac{1}{b}mx$ for $m \in \mathbb{N}$ in (4) we get

$$\begin{aligned}&\left\|g(x) - Ag\left(\frac{1}{a}(m+1)x\right) - Bg\left(-\frac{1}{b}mx\right)\right\| \\ &\quad\leq c\left(\left|\frac{1}{a}(m+1)\right|^p + \left|\frac{1}{b}m\right|^p\right)\|x\|^p, \quad x \in X\backslash\{0\}.\end{aligned} \tag{6}$$

Further put

$$\mathcal{T}_m\xi(x) := A\xi\left(\frac{1}{a}(m+1)x\right) + B\xi\left(-\frac{1}{b}mx\right), \qquad x \in X\backslash\{0\}, \xi \in Y^{X\backslash\{0\}},$$

$$\varepsilon_m(x) := c\left(\left|\frac{1}{a}(m+1)\right|^p + \left|\frac{1}{b}m\right|^p\right)\|x\|^p, \qquad x \in X\backslash\{0\}.$$

Then the inequality (6) takes the form

$$\|\mathcal{T}_m g(x) - g(x)\| \leq \varepsilon_m(x), \qquad x \in X\backslash\{0\}.$$

The operator

$$\Lambda_m\eta(x) := |A|\eta\left(\frac{1}{a}(m+1)x\right) + |B|\eta\left(-\frac{1}{b}mx\right), \qquad \eta \in {\mathbb{R}_+}^{X\backslash\{0\}}, x \in X\backslash\{0\}$$

has the form described in (H3) with $k = 2$ and $f_1(x) = \frac{1}{a}(m+1)x$, $f_2(x) = -\frac{1}{b}mx$, $L_1(x) = |A|$, $L_2(x) = |B|$ for $x \in X$. Moreover, for every $\xi, \mu \in Y^{X\backslash\{0\}}, x \in X\backslash\{0\}$

$$\begin{aligned}&\|\mathcal{T}_m\xi(x) - \mathcal{T}_m\mu(x)\| \\ &\quad= \left\|A\xi\left(\frac{1}{a}(m+1)x\right) + B\xi\left(-\frac{1}{b}mx\right) - A\mu\left(\frac{1}{a}(m+1)x\right) - B\mu\left(-\frac{1}{b}mx\right)\right\| \\ &\quad\leq |A|\left\|\xi\left(\frac{1}{a}(m+1)x\right) - \mu\left(\frac{1}{a}(m+1)x\right)\right\| + |B|\left\|\xi\left(-\frac{1}{b}mx\right) - \mu\left(-\frac{1}{b}mx\right)\right\| \\ &\quad= \sum_{i=1}^{2} L_i(x)\|\xi(f_i(x)) - \mu(f_i(x))\|.\end{aligned}$$

We see that there exists $m_0 \in \mathbb{N}$ such that

$$|A|\left|\frac{1}{a}(m+1)\right|^p + |B|\left|\frac{1}{b}m\right|^p < 1 \qquad \text{for } m \geq m_0.$$

Therefore

$$
\begin{aligned}
\varepsilon^*(x) &:= \sum_{n=0}^{\infty} \Lambda_m^n \varepsilon_m(x) \\
&= \sum_{n=0}^{\infty} c\left(\left|\frac{1}{a}(m+1)\right|^p + \left|\frac{1}{b}m\right|^p\right)\left(|A|\left|\frac{1}{a}(m+1)\right|^p + |B|\left|\frac{1}{b}m\right|^p\right)^n \|x\|^p \\
&= \frac{c(|\frac{1}{a}(m+1)|^p + |\frac{1}{b}m|^p)\|x\|^p}{1-|A||\frac{1}{a}(m+1)|^p - |B||\frac{1}{b}m|^p}, \qquad x\in X\backslash\{0\},\ m\geq m_0.
\end{aligned}
$$

Thus, according to Theorem 1, for each $m \geq m_0$ there exists a unique solution $G_m\colon X\backslash\{0\} \to Y$ of the equation

$$G_m(x) = AG_m\left(\frac{1}{a}(m+1)x\right) + BG_m\left(-\frac{1}{b}mx\right)$$

such that

$$\|g(x) - G_m(x)\| \leq \frac{c\left(|\frac{1}{a}(m+1)|^p + |\frac{1}{b}m|^p\right)\|x\|^p}{1-|A||\frac{1}{a}(m+1)|^p - |B||\frac{1}{b}m|^p}, \qquad x\in X\backslash\{0\}.$$

We show that

$$
\begin{aligned}
&\|\mathcal{T}_m^n g(ax+by) - A\mathcal{T}_m^n g(x) - B\mathcal{T}_m^n g(y)\| \\
&\quad\leq c\left(|A|\left|\frac{1}{a}(m+1)\right|^p + |B|\left|\frac{1}{b}m\right|^p\right)^n (\|x\|^p + \|y\|^p) \qquad (7)
\end{aligned}
$$

for every $x, y \in X\backslash\{0\}$, $n\in\mathbb{N}_0$.

If $n = 0$, then (7) is simply (4). So, take $r \in \mathbb{N}_0$ and suppose that (7) holds for $n = r$ and $x, y \in X\backslash\{0\}$. Then

$$
\begin{aligned}
&\|\mathcal{T}_m^{r+1} g(ax+by) - A\mathcal{T}_m^{r+1} g(x) - B\mathcal{T}_m^{r+1} g(y)\| \\
&= \Big\| A\mathcal{T}_m^r g\left(\frac{1}{a}(m+1)(ax+by)\right) + B\mathcal{T}_m^r g\left(-\frac{1}{b}m(ax+by)\right) \\
&\quad - A^2\mathcal{T}_m^r g\left(\frac{1}{a}(m+1)x\right) - AB\mathcal{T}_m^r g\left(-\frac{1}{b}mx\right) \\
&\quad - BA\mathcal{T}_m^r g\left(\frac{1}{a}(m+1)y\right) - B^2\mathcal{T}_m^r g\left(-\frac{1}{b}my\right)\Big\| \\
&\leq |A|\Big\|\mathcal{T}_m^r g\left(\frac{1}{a}(m+1)(ax+by)\right) - A\mathcal{T}_m^r g\left(\frac{1}{a}(m+1)x\right) \\
&\quad - B\mathcal{T}_m^r g\left(\frac{1}{a}(m+1)y\right)\Big\| \\
&\quad + |B|\Big\|\mathcal{T}_m^r g\left(-\frac{1}{b}m(ax+by)\right) - A\mathcal{T}_m^r g\left(-\frac{1}{b}mx\right) - B\mathcal{T}_m^r g\left(-\frac{1}{b}my\right)\Big\| \\
&\leq |A|c\left(|A|\left|\frac{1}{a}(m+1)\right|^p + |B|\left|\frac{1}{b}m\right|^p\right)^r \left(\left\|\frac{1}{a}(m+1)x\right\|^p + \left\|\frac{1}{a}(m+1)y\right\|^p\right)
\end{aligned}
$$

$$+ |B|c\left(|A|\left|\frac{1}{a}(m+1)\right|^p + |B|\left|\frac{1}{b}m\right|^p\right)^r \left(\left\|-\frac{1}{b}mx\right\|^p + \left\|-\frac{1}{b}my\right\|^p\right)$$
$$\leq c\left(|A|\left|\frac{1}{a}(m+1)\right|^p + |B|\left|\frac{1}{b}m\right|^p\right)^{r+1}(\|x\|^p + \|y\|^p).$$

Letting $n \to \infty$ in (7), we obtain that

$$G_m(ax+by) = AG_m(x) + BG_m(y), \qquad x, y \in X\backslash\{0\}.$$

So, we have a sequence $(G_m)_{m \geq m_0}$ of functions satisfying equation (5) such that

$$\|g(x) - G_m(x)\| \leq \frac{c\left(|\frac{1}{a}(m+1)|^p + |\frac{1}{b}m|^p\right)\|x\|^p}{1 - |A||\frac{1}{a}(m+1)|^p - |B||\frac{1}{b}m|^p}, \qquad x \in X\backslash\{0\}.$$

It follows, with $m \to \infty$, that g also satisfies (5) for $x, y \in X\backslash\{0\}$. □

Corolary 3. *Let X be a normed space over a field $\mathbb{F}$, Y be a Banach space over $\mathbb{K}$, $a, b \in \mathbb{F}\backslash\{0\}$, $A, B \in \mathbb{K}$, $A + B \neq 1$, $C \in Y$, $c \geq 0$, $p < 0$ and $f\colon X \to Y$ satisfy*

$$\|f(ax+by) - Af(x) - Bf(y) - C\| \leq c(\|x\|^p + \|y\|^p), \qquad x, y \in X\backslash\{0\}.$$

Then f satisfies the equation

$$f(ax+by) = Af(x) + Bf(y) + C, \qquad x, y \in X\backslash\{0\}.$$

Proof. Considering $g(x) = f(x) + \frac{C}{A+B-1}$, $x \in X$ we see that

$$\|g(ax+by) - Ag(x) - Bg(y)\| \leq c(\|x\|^p + \|y\|^p), \qquad x, y \in X\backslash\{0\}.$$

So, by Theorem 2 we have

$$g(ax+by) = Ag(x) + Bg(y), \qquad x, y \in X\backslash\{0\},$$

thus

$$f(ax+by) = Af(x) + Bf(y) + C, \qquad x, y \in X\backslash\{0\}.$$

□

Remark 4. If g satisfies (4) with $A = B = 0$, then, by (6) with $m \to \infty$, it follows that $g(x) = 0$ for $x \in X\backslash\{0\}$.

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